

A truly marginal deformation of $SL(2, R)$ in a null direction

Stefan Förste[‡]

Racah Institute of Physics

The Hebrew University, 91904 Jerusalem, Israel

Abstract

We perform a marginal deformation of the $SL(2, R)$ WZW model in a null direction. If we send the deformation parameter to infinity we obtain a linear dilaton background plus two free bosons. We show in addition that such a background can be obtained by a duality transformation of the undeformed WZW model. In the end we indicate how to generalize the given procedure.

One interesting observation in string theory is that theories with a completely different sigma model interpretation belong actually to the same conformal field theory. The reason for that is the existence of certain symmetries which change the background field configuration in a non trivial way. These symmetries are e.g. duality transformations [1] and $O(d, d, R)$ [2] transformations. Duality [4] and the discrete subgroup $O(d, d, Z)$ [3] connect sigma models belonging to the same conformal field theory whereas the $O(d, d, R)$ transforms between conformal backgrounds not necessarily corresponding to the same conformal field theory. (A review about these symmetries is given in [5].) Another interesting issue is that two different sigma models might be connected by a line of a truly marginal deformation [6, 7]. A way to find a truly marginal deformed model is to take a conformally invariant one parameter family of actions $S(\alpha)$ with a chiral and anti chiral symmetry. Then one has to show that a variation in α corresponds to a perturbation by a product of the chiral and anti chiral current, i.e. that the differential equation

$$\frac{\partial S(\alpha)}{\partial \alpha} \sim \int J(\alpha) \bar{J}(\alpha) \quad (1)$$

holds. Moreover one has to check that for α goes to zero one arrives at the original theory infinitesimally perturbed by a truly marginal operator. In [7] it was shown that deforming

[‡]e-mail: sforste@vms.huji.ac.il, Work supported by a grant of MINERVA

a WZW model of a group G leads in one end point of the deformation to a direct product of the gauged model $G/U(1)$ and a non compact $U(1)$, whereas in the other end $G/U(1)$ is replaced by it's dual. One can use the deformation of the $SU(2)$ WZW model given in [6, 7] in order to obtain a truly marginal deformed $SL(2, R)$ WZW model. There is an interesting difference between those two models. Deforming the $SL(2, R)$ in the $J_2\bar{J}_2$ direction one gets at one end of the deformation line the 2d euclidean black hole \times non compact $U(1)$ whereas deforming in the $J_3\bar{J}_3$ direction gives in the end the 2d lorentzian black hole \times non compact $U(1)$ [8]. In the other end the corresponding dual black hole backgrounds \times non compact $U(1)$ appear. (J_2 and J_3 are the $i\sigma_2$ and σ_3 components of the Kac-Moody current, respectively.) So, there is a difference between the deformations into a space like or time like direction. In the present paper we are going to consider a third possibility namely the deformation in a null direction.

We will give a deformed model satisfying (1) and the initial condition. As a check we will show that the deformed model is a conformal field theory to one loop order^a. Furthermore we will see that at the end points ($\alpha \rightarrow \pm\infty$) we get the dual of our original model.

We parametrize the $SL(2, R)$ as follows

$$g = g_- g_0 g_+, \quad (2)$$

with

$$g_- = \begin{pmatrix} 1 & 0 \\ \frac{x_-}{\sqrt{2}} & 1 \end{pmatrix} \quad g_0 = \begin{pmatrix} e^{\frac{x}{2}} & 0 \\ 0 & e^{-\frac{x}{2}} \end{pmatrix} \quad g_+ = \begin{pmatrix} 1 & \frac{x_+}{\sqrt{2}} \\ 0 & 1 \end{pmatrix}. \quad (3)$$

Then we obtain for the WZW model

$$S_{WZW} = \frac{k}{16\pi} \int tr (g^{-1} dg \wedge * g^{-1} dg) + \frac{ik}{24\pi} \int tr (g^{-1} dg^3) \quad (4)$$

the following action^b

$$S_{WZW} = \frac{k}{4\pi} \int d^2 z \left(\frac{1}{2} \partial x \bar{\partial} x + e^x \partial x_+ \bar{\partial} x_- \right), \quad (5)$$

with the conserved currents

$$J \sim e^x \partial x_+, \quad \bar{J} \sim e^x \bar{\partial} x_- . \quad (6)$$

Now we will show that the one parameter family of theories

$$S(\alpha) = \frac{k}{4\pi} \int d^2 z \left(\frac{1}{2} \partial x \bar{\partial} x + \frac{e^x}{1 - \alpha e^x} \partial x_+ \bar{\partial} x_- \right) \quad (7)$$

^aThis is actually not necessary since a truly marginal deformed exact CFT gives an exact CFT.

^bThe euclidean version can be obtained by taking x_+ and x_- as complex conjugated variables.

is the integrated marginal deformation of (5) with respect to $J\bar{J}$. (The action (7) can not be obtained by using the known $SU(2)$ deformation [6, 7]. It can rather be guessed, or a systematic way is to start with an infinitesimal perturbation resulting in an infinitesimal change in the currents. Repeating that procedure will result in a geometrical series which sums up to (7).)

From (7) we obtain the chiral currents

$$J(\alpha) \sim \frac{e^x}{1 - \alpha e^x} \partial x_+, \quad \bar{J}(\alpha) \sim \frac{e^x}{1 - \alpha e^x} \bar{\partial} x_-. \quad (8)$$

Differentiating $S(\alpha)$ with respect to α provides

$$\frac{\partial S(\alpha)}{\partial \alpha} \sim \int d^2 z J(\alpha) \bar{J}(\alpha). \quad (9)$$

For $\alpha = 0$ we obtain the original model. In the original theory the measures for the functional integral are defined with respect to the undeformed background. Introducing measures referring to the deformed background results in a jacobian which can be expressed by an additional dilaton term [1]. Taking that into account we obtain finally^c

$$S(\alpha) = \frac{k}{4\pi} \int d^2 z \left(\partial x \bar{\partial} x + \frac{e^x}{1 - \alpha e^x} \partial x_+ \bar{\partial} x_- \right) - \frac{1}{8\pi} \int d^2 z \sqrt{g} R \log(1 - \alpha e^x). \quad (10)$$

Since the dilaton does not contain x_{\pm} it does not change the definition of the currents (8). It's variation with α cancels against the variation of the measure [7]. Hence (9) still holds. Modifying the consideration given in the appendix of [7] slightly one can convince himself that (10) is the only solution satisfying (1) and the initial condition. As a test we checked the one loop conformal invariance. We found that (10) is really a conformal field theory with central charge

$$c = 3 + \frac{12}{k} + o\left(\frac{1}{k^2}\right) \quad (11)$$

at one loop level. The central charge does not depend on α and coincides with the central charge of the original model [9]

$$c = \frac{3k}{4 - k}. \quad (12)$$

The deformation parameter α can run from minus infinity to plus infinity giving $SL(2, R)$ at zero. In order to check what happens at the borders we set $\alpha = \frac{1}{\epsilon}$ rescale

$$x_{\pm} \rightarrow \frac{1}{\sqrt{\epsilon}} x_{\pm}$$

and send ϵ to zero. That results in

$$S(\infty) = \frac{k}{4\pi} \int d^2 z \left(\frac{1}{2} \partial x \bar{\partial} x - \partial x_+ \bar{\partial} x_- \right) - \frac{1}{8\pi} \int d^2 z \sqrt{g} R x. \quad (13)$$

^cIn [12] this background was obtained as a gauged $SL(2, R) \times R/R$ WZW model.

This is a theory consisting of a linear dilaton background and two free bosons. (At minus infinity we get a plus sign in front of $\partial x_+ \bar{\partial} x_-$.) The linear dilaton background appears also when we gauge the $SL(2, R)$ with respect to \bar{J} from the left and with respect to J from the right [10]. So, at the end point of the deformation the deformed theory decouples into a direct product of the gauged theory and two bosons. This is quite similar to the case discussed in [6, 7] where at the end points of the deformation direct products occur which contain the axial or vector gauged WZW model.

Now we are going to show that $S(0)$ and $S(\infty)$ describe equivalent quantum theories. As a first step we gauge the translational symmetry of x_- in (5) and write down the identity

$$\int \mathcal{D}x_+ \mathcal{D}x_- e^{-S_{WZW}} = \int \mathcal{D}A \mathcal{D}\bar{A} \mathcal{D}\lambda \mathcal{D}x_+ \mathcal{D}x_- \det(\partial \bar{\partial}) e^{-S(A, \bar{A})}, \quad (14)$$

where

$$S(A, \bar{A}) = \frac{k}{4\pi} \int d^2z \left\{ \frac{1}{2} \partial x \bar{\partial} x + e^x \partial x_+ \bar{\partial} x_- + \bar{A} e^x \partial x_+ + \lambda (\partial \bar{A} - \bar{\partial} A) \right\}. \quad (15)$$

The identity (14) is known from duality transformations [4]. One can check (14) by first performing the λ integration and then using the resulting δ -function to express the gauge fields as a pure gauge, (the determinant $\det(\partial \bar{\partial})$ cancels when one performs a field redefinition in such a way that there are no derivatives in the argument of the δ -function). Being a pure gauge field \bar{A} can be absorbed in $\bar{\partial} x_-$. In order to obtain a different background we do not perform the λ integration, instead we integrate out \bar{A} which provides a δ -function,

$$\delta(e^x \partial x_+ - \partial \lambda).$$

We use that δ -function to integrate out x_+ and get

$$\int \mathcal{D}x_+ \mathcal{D}x_- e^{-S_{WZW}} = \int \mathcal{D}A \mathcal{D}\lambda \mathcal{D}x_- \det(\bar{\partial}) e^{-S_1}, \quad (16)$$

where S_1 is given by

$$S_1 = \frac{k}{4\pi} \int d^2z \left(\frac{1}{2} \partial x \bar{\partial} x + \partial \lambda \bar{\partial} x_- + A \bar{\partial} \lambda \right) - \frac{1}{8\pi} \int d^2z \sqrt{g} R x. \quad (17)$$

The linear dilaton in (17) arises from the determinant

$$(\det e^x \partial)^{-1} = (\det e^x \det \partial)^{-1}.$$

Now integrating over λ gives

$$\delta(\bar{\partial} A + \partial \bar{\partial} x_-).$$

Again we perform a field redefinition in such a way that there are no derivatives in the δ -function. Hence the A, x_- integrations lead just to

$$\det \bar{\partial}^{-1} (\det \bar{\partial} \partial)^{-1}.$$

The first factor cancels with $\det \bar{\partial}$ in (16) and the second factor can be expressed as a functional integral of two free bosons. Hence we get finally

$$\int \mathcal{D}x_+ \mathcal{D}x_- e^{-S_{WZW}} = \int \mathcal{D}y_+ \mathcal{D}y_- e^{-\hat{S}}, \quad (18)$$

with

$$\hat{S} = \frac{k}{4\pi} \int d^2z \frac{1}{2} \partial x \bar{\partial} x - \frac{1}{8\pi} \int d^2z \sqrt{g} R x + \frac{1}{4\pi} \int d^2z \partial y_+ \bar{\partial} y_- \quad (19)$$

which is exactly the theory we obtained by an infinite marginal deformation^d. So, the $\alpha = 0$ and the $\alpha = \infty$ points of our family of exact conformal field theories (10) are connected by a duality transformation. In fact, one can do this duality transformation at any finite α and gets always (19). That shows that the whole deformation line consists of equivalent conformal field theories up to global issues which were not taken into account. However, global issues might be important, e.g. neglecting them one gets the result that the conformal field theory of a free boson compactified on a circle is dual to a non compact free boson independent of the radius of the circle. (The more familiar duality transformation replacing the metric by it's inverse can be obtained by gauging the translational invariance of x_{\pm} simultaneously.)

So, in difference to the $SU(2)$ WZW model the $SL(2, R)$ WZW model allows (at least) for three different truly marginal deformations. In the end points we get either a direct product of the 2d lorentzian black hole and a free boson, the 2d euclidean black hole and a free boson, or a linear dilaton background and two free bosons. That shows that the $c = 1$ non critical string and 2d black hole physics are connected (cf.[11]) though not equivalent (since it is not possible to rotate a null direction into a time or space like direction).

Finally we remark that the presented considerations can be generalized for any CFT of the form

$$S = \frac{k}{4\pi} \int d^2z \left(E_{\mu\nu}(r) \partial r^\mu \bar{\partial} r^\nu + F(r) \partial x_+ \bar{\partial} x_- \right) + \frac{1}{4\pi} \int d^2z \sqrt{g} R \Phi(r). \quad (20)$$

The deformed model is^e

$$\begin{aligned} S(\alpha) = & \frac{k}{4\pi} \int d^2z \left(E_{\mu\nu}(r) \partial r^\mu \bar{\partial} r^\nu + \frac{F(r)}{1-\alpha F(r)} \partial x_+ \bar{\partial} x_- \right) \\ & + \frac{1}{4\pi} \int d^2z \sqrt{g} R \left(\Phi(r) - \frac{1}{2} \log(1 - \alpha F(r)) \right). \end{aligned} \quad (21)$$

With

$$J(\alpha) \sim \frac{F(r)}{1 - \alpha F(r)} \partial x_+, \quad \bar{J}(\alpha) \sim \frac{F(r)}{1 - \alpha F(r)} \bar{\partial} x_- \quad (22)$$

^dSince we did not consider global effects when calculating the dual action it might differ from $S(\infty)$ topologically.

^eThat this deformation respects the conformal invariance is shown in [12] in a different way.

it is easy to check that

$$\frac{\partial S}{\partial \alpha} \sim \int J(\alpha) \bar{J}(\alpha). \quad (23)$$

Models of the form (20) have been considered in [10] in the connection with gauging by null subgroups. A detailed application of our marginal deformation to the more general case (20) will be done soon. Forthcoming work will also address problems left open here, e.g. global issues, or whether the discussed three directions are the only non equivalent ones, (in that context it might be interesting to study a deformation where for example the chiral current belongs to a null direction and the anti chiral current belongs to a space or time like direction).

I would like to thank Amit Giveon, Eliezer Rabinovici and Gautam Sengupta for very useful discussions. Moreover I would like to thank Arkady Tseytlin for comments.

References

- [1] T. Buscher, Phys. Lett. **B194**(1987)59;
T. Buscher, Phys. Lett. **B201**(1988)466
- [2] K. A. Meissner, G. Veneziano, Phys. Lett. **B267** (1991) 33;
M. Gasperini, J. Maharana, G. Veneziano, Phys. Lett. **B272**(1991)277;
A. Sen, Phys. Lett. **B271**(1991)33
- [3] A. Giveon, M. Rocek, Nucl. Phys. **B380**(1992)128
- [4] M. Rocek, E. Verlinde, Nucl. Phys. **B373**(1992)630
- [5] A. Giveon, M. Porrati, E. Rabinovici, RI-1-94,NYU-TH.94/01/01,(hep-th/9401139)
- [6] S. F. Hassan, A. Sen, Nucl. Phys. **B405**(1993)143
- [7] A. Giveon, E. Kiritsis, Nucl. Phys. **B411**(1994)487
- [8] S. Elitzur, A. Forge, E. Rabinovici, Nucl. Phys. **B359**(1991)581;
E. Witten, Phys. Rev. **D44**(1991)314
- [9] V. G. Knizhnik, A. B. Zamolodchikov, Nucl. Phys. **B247**(1984)83
- [10] C. Klimcik, A. A. Tseytlin, preprint Imperial/93-94/17 (hep-th/9402120)
- [11] A. Alekseev, S. Shatashvili, Nucl. Phys. **B323**(1989)719;
M. Bershadsky, H. Ooguri, Comm. Math. Phys. **126**(1989)49;
P. Forgacs, A. Wipf, J. Balog, L. Feher, L. O’Raifeartaigh, Phys. Lett. **B227**(1989)214;
R. Dijkgraaf, H. Verlinde, E. Verlinde, Nucl. Phys. **B371**(1992)269
- [12] G. T. Horowitz, A. A. Tseytlin, Imperial/TP/93-94/38, NI94004,(hep-th/9406067)